EE 411: Circuit Theory

Response of First Order RL and RC Circuits

Outline

- The Step Response of an RL Circuit
- The Natural Response of an RL Circuit
- The Step Response of an RC Circuit
- The Natural Response of an RC Circuit
- The General Solution Form

Focus

- Circuits that consist only of sources, resistors and either inductors or capacitors
  - Those configurations are called RL and RC circuits
- We will consider the currents and voltages that arise due to the sudden application (or removal) of a constant (dc) voltage or current source

Examples

- [Diagrams showing RL and RC circuits with initial conditions and step functions]
Step Response to an RL Circuit

For \( t > 0 \) (after the switch is closed)

KVL: \( v_s(t) + v_0(t) - V_s = 0 \)

\[
L \frac{di(t)}{dt} + R i(t) = V_s
\]

forcing function

First order ordinary differential equation

Solving the DE: Deriving \( i(t) \)

\[
L \frac{di(t)}{dt} + R i(t) = V_s
\]

\[
\frac{di(t)}{dt} = -\frac{R}{L} i(t) + \frac{V_s}{L}
\]

\[= \frac{-R}{L} \left( i(t) - \frac{V_s}{R} \right) dt \]

\[= \frac{-R}{L} \frac{V_s}{R} dt \]

\[\begin{align*}
\frac{di(t)}{i(t) - \frac{V_s}{R}} &= \frac{-R}{L} \\
\ln |i(t) - \frac{V_s}{R}| &= -\frac{R}{L} t + C \\
\ln |i(t) - \frac{V_s}{R}| &= \ln |e^{\frac{-R}{L} t}|
\end{align*}\]

\[\begin{align*}
\ln |i(t) - \frac{V_s}{R}| &= \ln |e^{\frac{-R}{L} t}|
\Rightarrow |i(t) - \frac{V_s}{R}| &= e^{\frac{-R}{L} t}
\end{align*}\]

Solving the DE: Deriving \( i(t) \) (cont)

\[\int_{i(0)}^{i(t)} \frac{di}{i} = \int_{0}^{t} \frac{-R}{L} dt \]

\[\begin{align*}
\ln |i(t) - \frac{V_s}{R}| &= -\frac{R}{L} t + C \\
\ln |i(t) - \frac{V_s}{R}| &= \ln |e^{\frac{-R}{L} t}|
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\end{align*}\]

Step Response of RL Circuit (cont)

\[i(t) = \frac{V_s}{R} + \left( i(0) - \frac{V_s}{R} \right) e^{\frac{-t}{R \tau}}\]

\[\begin{align*}
i(t) &= \frac{V_s}{R} + \left( i(0) - \frac{V_s}{R} \right) e^{\frac{-t}{R \tau}} \\
i(t) &= \frac{V_s}{R} - \frac{V_s}{R} e^{\frac{-t}{R \tau}} \\
i(t) &= \frac{V_s}{R} \left( 1 - e^{\frac{-t}{R \tau}} \right)
\end{align*}\]

where \( \tau = \frac{R}{L} \) = [sec] \( \Rightarrow \) time constant or characteristic time \( (1 \text{ H} = 1 \text{V/A}) \)
**Step Response of an RL Circuit**

For \( t > 0 \) (after the switch is closed)

KVL: \( v_L(t) + v_R(t) - V_S = 0 \)

\[
L \frac{di(t)}{dt} + R i(t) = V_S
\]

forcing function

First order circuit \( \rightarrow \) First order ordinary differential equation

complete solution \( i(t) = \) “homogeneous solution” + “particular solution”

(\textit{step response}): response of \( i(t) \) due to a sudden application of a constant voltage or current

(\textit{natural response}): response when the forcing function is 0 (i.e., due to the nature of the circuit)

(\textit{forced response})
Step/Complete Response of an RL Circuit

\[ i(t) = \frac{V_s}{R} + \left[ i(0^+) - \left( \frac{V_s}{R} \right) \right] e^{-t/\tau} \]

Complete solution \( i(t) \) = “homogeneous solution” + “particular solution”

- Transient response
- Steady state - constant (for constant source \( V \))

Natural Response of an RL Circuit

\[ i(t) = \left( i(0^+) - \left( \frac{V_s}{R} \right) \right) e^{-t/\tau} \]

- Transient Solution (natural response)
- Steady state solution

**Natural** Response of an RL Circuit

\[ i(t) = \frac{V_s}{R} e^{-t/\tau} \]

- Transient Solution
- Steady state solution

**Natural** Response of an RL Circuit

\[ i(t) = \frac{V_s}{R} e^{-t/\tau} \]

- Transient Solution
- Steady state solution

Time constant (\( \tau \)) is the same as before, since RL circuit is identical...
**RL Circuit: Example 2**

Eventually, \( L \) behaves as a short circuit.

- \( i_L(t) = I_S (1 - e^{-\frac{t}{\tau}}) \) (Assume)
- \( \frac{V_S}{R} (1 - e^{-\frac{t}{\tau}}) \) (Th)

**Step Response for an RC Circuit**

For \( t > 0 \) (after the switch is closed)

KVL: \( \nu(t) + v_C(t) - V_S = 0 \)

\[ RC \frac{dv_C(t)}{dt} + v_C(t) = V_S \]

First order ordinary differential equation.

**Outline**

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**Step Response for an RC Circuit**

\[ \tau = RC \quad \text{(dual of } \tau = LR) \]

- Steady state solution or particular solution
- Transient Solution (natural response)

**Step Response**

\[ v_C(t) = V_S (1 - e^{-\frac{t}{\tau}}) \]
Step Response for an RC Circuit

\[ i(t) = \frac{V_s - V_c(1 - e^{-\frac{t}{\tau}})}{R} = \frac{(V_s/R)e^{-\frac{t}{\tau}}}{t} \quad \text{or } C \times \text{derivative of } v_c(t) \]

\[ v_c(t) = V_s(1 - e^{-\frac{t}{\tau}}) \]

\[ \tau = RC \]

Step Response for an RC Circuit, Example 2

\[ v_c(t) = V_2 + V_3(1 - e^{-\frac{t}{\tau}}) \]

\[ V_s = V_1 \cdot V_2 \]

\[ v_c(t) = v_c(\infty) + \left( v_c(0^+) - v_c(\infty) \right)e^{-\frac{t}{\tau}} \]

\[ t \to 0; \quad v_c(t) \to V_1 - V_2 \quad V_2 = V_1 \cdot V_2 \]

\[ t \to \infty; \quad v_c(t) \to V_2 + V_3 = V_1 \]

General Solution Form

1st order circuit always gives a Differential Equation in the form of

\[ \frac{dX}{dt} + \frac{X}{\tau} = \text{constant} \]

\[ (X: \text{response, } v \text{ or } i) \]

Complete response has the form:

\[ X(t) = X_0 + (X_\infty - X_0)e^{-\frac{t}{\tau}} \]

* For RC \( \tau = R_C \cdot C \) (\( R_C \) is equivalent resistance across the terminals of the capacitance, with independent sources suppressed)

* For RL \( \tau = L/R \) (\( R \) is equivalent resistance across the terminals of the inductance, with independent sources suppressed)